MULTIMEDIA UNIVERSITY

STUDENT IDENTIFICATION NO

MULTIMEDIA UNIVERSITY

FINAL EXAMINATION

TRIMESTER 2, 2016/2017 SESSION

BMS2024 - ADVANCED MANAGERIAL STATISTICS

(All Sections / Groups)

4 March 2017
2.30pm – 4.30pm
(2 Hours)

INSTRUCTIONS TO STUDENTS

1. This question paper consists of 14 pages **excluding** the cover page.
2. This question paper consists of **FOUR** structured questions. Attempt **ALL** questions.
3. Students are allowed to use non-programmable scientific calculators with no restrictions.
4. A formulae list and statistical tables are attached at the end of the question paper.
5. Use **pen** to write the answers in the answer booklet provided.

QUESTION 1

The electrical device in a semi-conductor plant have been found to have a mean lifetime of 1030.00 hours with a standard deviation of 90.00 hours. The plant engineer has been approached by a sales executive of Electro-Last, a company that makes a gadget intended to increase the lifetime of the electrical device. The engineer assumes that, even if the gadget is fitted the average lifetime of the electrical device might not be any greater than the current 1030.00 hours. To verify his assumption, the engineer tests 40 electrical devices which were fitted with the gadget and finds their mean lifetime to be 1061.60 hours.

Based on the above, answer the following:

a) Establish the null and alternative hypotheses. [2 marks]

b) Compute the test-statistic and the p-value of the test at the 5 per cent level of significance. Explain the result. [5 marks]

c) State and discuss Type I error. [3 marks]

d) State and discuss Type II error. [3 marks]

e) Define the power of the statistical test. [3 marks]

f) For a given sample size, if the probability of the Type I error is decreased, what is the effect on the probability of a Type II error? [3 marks]

g) If the sample size is increased, what is the effect on the Type II error? [3 marks]

h) Can both errors be eliminated? Justify your answer. [3 marks]

[Total: 25 Marks]

QUESTION 2

A medium sized fertilizer plant is evaluating two accounting software packages for use in its accounts department. The two software packages evaluated are, one called Account Super and the other, Account Tally. The plant accountant has selected 12 different computing activities that are typical of the tasks that such a package would have to perform, and then recorded the number of seconds each package takes to complete each task. At the 0.10 level of significance, can we conclude that the median difference for the population of such tasks is zero? The details shown below:

Continued...

Time Required for Software Package

Computing Task	Account Super	Account Tally
A	24	23
B	17	20
C	22	18
D	24	21
E	38	42
F	31	36
G	15	22
H	37	40
I	18	26
J	16	16
K	29	35
L	20	26

a) Do the data exhibit related or unrelated samples? Explain. [3 marks]

b) Between parametric and nonparametric testing, which statistical testing method is more suitable? Explain. [3 marks]

c) Conduct an appropriate statistical testing method. Be sure to show all steps and calculations. [19 marks]

[Total: 25 Marks]

QUESTION 3

Arctic Enterprise produces Polar, a brand of liquid laundry detergent. In order to manage its inventory more effectively and make revenue projections, the company would like to better predict demand for its product. To develop a predictive model, the company has gathered data in relation to demand for its large sized bottles over the last 30 sales periods.

The variables that would be incorporated into the predictive model will be the retail price of the product, the average industry price of competitors' similar detergents, and the company's advertising expenditure. The data are analysed and the summary output of the analysis is shown below:

Continued...

Regression Statistics	
Multiple R	0.9453
R Square	0.8936
Adjusted R Square	0.8813
Standard Error	0.2347
Observations	30

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	12.0268	4.0089	72.7973	8.88341E-13
Residual	26	1.4318	0.0551		
Total	29	13.4586			

	Coefficients	*Std Error*	*t Stat*	*P-value*
Intercept	7.5891	2.4450	3.1039	0.00457
Retail Price	–2.3577	0.6379	–3.6958	0.00103
Industry Price	1.6122	0.2954	5.4586	0.00001
Advertising Expenditure	0.5012	0.1259	3.9814	0.00049

a) What is multiple linear regression analysis? Explain. [5 marks]

b) State the multiple linear regression equation for the above data. [5 marks]

c) Compare the values of the coefficient of determination with the adjusted coefficient of determination. Explain why the values differ. [5 marks]

d) At the 1 percent level of significance, test if the model is valid. Use the p-value approach. [5 marks]

e) At the 1 percent level of significance, test if each independent variable have a significant contribution to the model. Use the p-value approach. [5 marks]

[Total: 25 Marks]

Continued...

QUESTION 4

Internet security is a serious issue in the digital era. The prevalence of spam messages is the price we pay for being able to easily communicate by e-mail. In a preliminary study on spam messages received in an insurance firm, company directors, executives, and administrative staff were randomly sampled. Each person was asked to count the number of spam messages received that day. The data in the table below represents the number of spam received by the three different categories of staff in the firm. Using $\alpha = 0.05$, analyse the data using the correct statistical procedure.

Directors	Executives	Admin. Staff
7	5	12
4	9	4
0	12	5
3	16	18
18	10	15

Summary Output

Groups	Count	Sum	Mean	Variance
Directors	5	32	6.4	48.3
Executives	5	52	10.4	16.3
Admin. Staff	5	54	10.8	37.7

ANOVA

Source of Variation	SS	df	MS	F
Among Groups	59.2	2	29.6	0.87
Within Groups	409.2	12	34.1	
Total	468.2	14		

a) State the required conditions or assumptions for the ANOVA test to be conducted? [3 marks]

b) What kind of ANOVA test will be appropriate for this study? What are the dependent variable and the independent variables (factors) that needs to be identified? Conduct an appropriate statistical procedure in testing the difference in the mean number of spam received by the three different categories of staff? [10 marks]

c) Conduct the Tukey-Kramer post-hoc test to examine if there is a significant change in the number of spam received by the three different categories of staff. (NB: The value of Qu in the Critical Range formula is 3.08) [12 marks]

[Total: 25 marks]

End of Paper

STATISTICAL FORMULAE

A. DESCRIPTIVE STATISTICS

Sample Mean = $\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$ Sample Standard Deviation = s = $\sqrt{\dfrac{\sum_{i=1}^{n} X_i^2}{n-1} - \dfrac{\left(\sum_{i=1}^{n} X_i\right)^2}{n(n-1)}}$

where n = *number of observations*
X_i = *the* i^{th} *observation of the data*

B. HYPOTHESIS TESTING

Types of Error

Type I Error = α= P(Rejecting H_0 | H_0 is true) where, Confidence Interval = 1 - α

Type II Error = β= P(Not Rejecting H_0 | H_0 is false)

One Sample Mean Test

σ Known	**σ Unknown**
$z = \dfrac{\overline{x} - \mu}{\sigma/\sqrt{n}}$	$t = \dfrac{\overline{x} - \mu}{s/\sqrt{n}}$

Two Sample Mean Test

Comparing Means for Two Independent Populations

[Standard Deviation (σ) Known]

$$z = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

[Standard Deviation (σ) Not Known]

$$t = \frac{(\overline{x_1} - \overline{x_2}) - (\mu_1 - \mu_2)}{\sqrt{S_p^2\left(1/n_1 + 1/n_2\right)}}$$

where $S_p^2 = \dfrac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 + n_2 - 2)}$

Two Sample Mean Test

Comparing Means for Two Paired Populations

$$t = \frac{\left(\overline{D} - \mu_D\right)}{S_D/\sqrt{n}} \quad \text{where } \overline{D} = \frac{\sum_{i=1}^{n} D_i}{n} \quad \text{and } S_D = \sqrt{\frac{\sum_{i=1}^{n} D_i^2}{n-1} - \frac{\left(\sum_{i=1}^{n} D_i\right)^2}{n(n-1)}}$$

Non-Parametric Analysis

Wilcoxon Rank Sum Test

$$Z = \frac{(T_1 - \mu_{T_1})}{\sigma_{T_1}} \quad \text{where}$$

$$\mu_{T1} = \frac{n_1(n+1)}{2} \quad \text{and}$$

$$\sigma_{T_1} = \sqrt{\frac{n_1 n_2 (n+1)}{12}} \quad \text{where } n = n_1 + n_2$$

Wilcoxon Signed Rank Sum Test

$$Z = \frac{(T_+ - \mu_{T_+})}{\sigma_{T_+}} \quad \text{where}$$

$$\mu_{T+} = \frac{n(n+1)}{4} \quad \text{and}$$

$$\sigma_{T_+} = \sqrt{\frac{n(n+1)(2n+1)}{24}}$$

Kruskal-Wallis Rank Test

$$H = \left[\frac{12}{n(n+1)} \sum_{j=1}^{c} \frac{T_j^2}{n_j} \right] - 3(n+1) \quad \text{where the critical value is } \chi^2 \text{ with } df = c - 1$$

Check ranking sum: $\sum T_j = n(n+1)/2$

Chi-Square Test

$$\chi^2 = \sum^{n} \frac{(O-E)^2}{E}$$

where O = Frequency of Observed Values and E = Frequency of Expected Values

with df = c – 1 where c = number of categories
or
with df = (r – 1)(c – 1) where r = number of rows and c = number of columns

C. ANALYSIS OF VARIANCE (ANOVA)

One-Way ANOVA

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F-statistic
Among Groups	*c - 1*	SSA	MSA = SSA/*c-1*	MSA/MSW
Within Groups	*n - c*	SSW	MSW = SSW/*n-c*	
Total	*n - 1*	SST		

$$SST = \sum_{j=1}^{c} \sum_{i=1}^{n_j} \left(X_{ij} - \overline{\overline{X}} \right)^2 \quad \textit{or alternative formula:} \quad SST = \left(\sum_{j=1}^{c} \sum_{i=1}^{n_j} X_{ij}^2 \right) - \frac{\left(\sum_{j=1}^{c} \sum_{i=1}^{n_j} X_{ij} \right)^2}{n}$$

$$SSA = \sum_{j=1}^{c} n_j \left(\overline{X}_j - \overline{\overline{X}} \right)^2 \quad \textit{and SSW = SST - SSA}$$

where n = number of observations, c = number of groups and $\overline{\overline{X}}$ *= overall mean*

Tukey-Kramer Procedure

$$\text{Critical Range} = Q_U \sqrt{\frac{MSW}{2}\left[\frac{1}{n_i}+\frac{1}{n_j}\right]}$$

where Q_u = the upper tail critical value from a Studentized Range Distribution having (c) degrees of freedom in the numerator and (n – c) degrees of freedom in the denominator at a given level of significance

Two-Way ANOVA

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F-statistic
A	$r-1$	SSA	MSA = SSA/$(r-1)$	MSA / MSE
B	$c-1$	SSB	MSB = SSB/$(c-1)$	MSB / MSE
AB	$(r-1)(c-1)$	SSAB	MSAB = SSAB/$(r-1)(c-1)$	MSAB / MSE
Error	$rc\,(n-1)$	SSE	MSE = SSE/$rc(n'-1)$	
Total	$n-1$	SST		

where,

Factor A levels are represented by the rows and Factor B levels are represented by the columns and
n = number of observations
c = number of columns
r = number of rows
n' = number of replicates

$$SST = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n'}\left(X_{ijk} - \overline{\overline{X}}\right)^2 \qquad SSA = cn'\sum_{i=1}^{r}\left(\overline{X}_i - \overline{\overline{X}}\right)^2$$

$$SSB = rn'\sum_{j=1}^{c}\left(\overline{X}j - \overline{\overline{X}}\right)^2 \qquad \text{where } \overline{\overline{X}} = \textit{overall mean}$$

$$SSE = (n'-1)\left[S_1^2 + S_2^2 + \ldots\ldots + S_k^2\right] \qquad \text{where } S_i^2 = \text{variance of each block}$$

D. REGRESSION ANALYSIS

Multiple Linear Regression

Population Model: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots\ldots + \beta_k X_k + \varepsilon$

Sample Model: $\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \ldots\ldots + b_k x_k + e$

$$\text{Adjusted R-Square} = 1 - \left[\frac{(1-R^2)(n-1)}{(n-p-1)}\right]$$ where p = number of independent/predictor variables

ANOVA Table for Regression			
Source	**Degrees of Freedom**	**Sum of Squares**	**Mean Squares**
Regression	p	SSR	MSR = SSR/p
Error/Residual	$n-p-1$	SSE	MSE = SSE/$(n-p-1)$
Total	$n-1$	SST	

Test Statistic for Significance of the Overall Regression Model

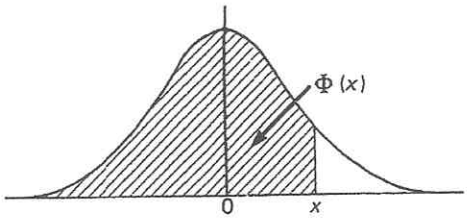
F = MSR/MSE

Test Statistic for Significance of Each Predictor Variable

$t_i = \dfrac{b_i}{S_{b_i}}$ and the critical value $= \pm t_{\alpha/2,(n-p-1)}$

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

The function tabulated is $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean and unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with zero mean and unit variance is symmetric about zero.



x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0·00	0·5000	0·40	0·6554	0·80	0·7881	1·20	0·8849	1·60	0·9452	2·00	0·97725
·01	·5040	·41	·6591	·81	·7910	·21	·8869	·61	·9463	·01	·97778
·02	·5080	·42	·6628	·82	·7939	·22	·8888	·62	·9474	·02	·97831
·03	·5120	·43	·6664	·83	·7967	·23	·8907	·63	·9484	·03	·97882
·04	·5160	·44	·6700	·84	·7995	·24	·8925	·64	·9495	·04	·97932
0·05	0·5199	0·45	0·6736	0·85	0·8023	1·25	0·8944	1·65	0·9505	2·05	0·97982
·06	·5239	·46	·6772	·86	·8051	·26	·8962	·66	·9515	·06	·98030
·07	·5279	·47	·6808	·87	·8078	·27	·8980	·67	·9525	·07	·98077
·08	·5319	·48	·6844	·88	·8106	·28	·8997	·68	·9535	·08	·98124
·09	·5359	·49	·6879	·89	·8133	·29	·9015	·69	·9545	·09	·98169
0·10	0·5398	0·50	0·6915	0·90	0·8159	1·30	0·9032	1·70	0·9554	2·10	0·98214
·11	·5438	·51	·6950	·91	·8186	·31	·9049	·71	·9564	·11	·98257
·12	·5478	·52	·6985	·92	·8212	·32	·9066	·72	·9573	·12	·98300
·13	·5517	·53	·7019	·93	·8238	·33	·9082	·73	·9582	·13	·98341
·14	·5557	·54	·7054	·94	·8264	·34	·9099	·74	·9591	·14	·98382
0·15	0·5596	0·55	0·7088	0·95	0·8289	1·35	0·9115	1·75	0·9599	2·15	0·98422
·16	·5636	·56	·7123	·96	·8315	·36	·9131	·76	·9608	·16	·98461
·17	·5675	·57	·7157	·97	·8340	·37	·9147	·77	·9616	·17	·98500
·18	·5714	·58	·7190	·98	·8365	·38	·9162	·78	·9625	·18	·98537
·19	·5753	·59	·7224	·99	·8389	·39	·9177	·79	·9633	·19	·98574
0·20	0·5793	0·60	0·7257	1·00	0·8413	1·40	0·9192	1·80	0·9641	2·20	0·98610
·21	·5832	·61	·7291	·01	·8438	·41	·9207	·81	·9649	·21	·98645
·22	·5871	·62	·7324	·02	·8461	·42	·9222	·82	·9656	·22	·98679
·23	·5910	·63	·7357	·03	·8485	·43	·9236	·83	·9664	·23	·98713
·24	·5948	·64	·7389	·04	·8508	·44	·9251	·84	·9671	·24	·98745
0·25	0·5987	0·65	0·7422	1·05	0·8531	1·45	0·9265	1·85	0·9678	2·25	0·98778
·26	·6026	·66	·7454	·06	·8554	·46	·9279	·86	·9686	·26	·98809
·27	·6064	·67	·7486	·07	·8577	·47	·9292	·87	·9693	·27	·98840
·28	·6103	·68	·7517	·08	·8599	·48	·9306	·88	·9699	·28	·98870
·29	·6141	·69	·7549	·09	·8621	·49	·9319	·89	·9706	·29	·98899
0·30	0·6179	0·70	0·7580	1·10	0·8643	1·50	0·9332	1·90	0·9713	2·30	0·98928
·31	·6217	·71	·7611	·11	·8665	·51	·9345	·91	·9719	·31	·98956
·32	·6255	·72	·7642	·12	·8686	·52	·9357	·92	·9726	·32	·98983
·33	·6293	·73	·7673	·13	·8708	·53	·9370	·93	·9732	·33	·99010
·34	·6331	·74	·7704	·14	·8729	·54	·9382	·94	·9738	·34	·99036
0·35	0·6368	0·75	0·7734	1·15	0·8749	1·55	0·9394	1·95	0·9744	2·35	0·99061
·36	·6406	·76	·7764	·16	·8770	·56	·9406	·96	·9750	·36	·99086
·37	·6443	·77	·7794	·17	·8790	·57	·9418	·97	·9756	·37	·99111
·38	·6480	·78	·7823	·18	·8810	·58	·9429	·98	·9761	·38	·99134
·39	·6517	·79	·7852	·19	·8830	·59	·9441	·99	·9767	·39	·99158
0·40	0·6554	0·80	0·7881	1·20	0·8849	1·60	0·9452	2·00	0·9772	2·40	0·99180

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
2·40	0·99180	2·55	0·99461	2·70	0·99653	2·85	0·99781	3·00	0·99865	3·15	0·99918
·41	·99202	·56	·99477	·71	·99664	·86	·99788	·01	·99869	·16	·99921
·42	·99224	·57	·99492	·72	·99674	·87	·99795	·02	·99874	·17	·99924
·43	·99245	·58	·99506	·73	·99683	·88	·99801	·03	·99878	·18	·99926
·44	·99266	·59	·99520	·74	·99693	·89	·99807	·04	·99882	·19	·99929
2·45	0·99286	2·60	0·99534	2·75	0·99702	2·90	0·99813	3·05	0·99886	3·20	0·99931
·46	·99305	·61	·99547	·76	·99711	·91	·99819	·06	·99889	·21	·99934
·47	·99324	·62	·99560	·77	·99720	·92	·99825	·07	·99893	·22	·99936
·48	·99343	·63	·99573	·78	·99728	·93	·99831	·08	·99896	·23	·99938
·49	·99361	·64	·99585	·79	·99736	·94	·99836	·09	·99900	·24	·99940
2·50	0·99379	2·65	0·99598	2·80	0·99744	2·95	0·99841	3·10	0·99903	3·25	0·99942
·51	·99396	·66	·99609	·81	·99752	·96	·99846	·11	·99906	·26	·99944
·52	·99413	·67	·99621	·82	·99760	·97	·99851	·12	·99910	·27	·99946
·53	·99430	·68	·99632	·83	·99767	·98	·99856	·13	·99913	·28	·99948
·54	·99446	·69	·99643	·84	·99774	·99	·99861	·14	·99916	·29	·99950
2·55	0·99461	2·70	0·99653	2·85	0·99781	3·00	0·99865	3·15	0·99918	3·30	0·99952

The critical table below gives on the left the range of values of x for which $\Phi(x)$ takes the value on the right, correct to the last figure given; in critical cases, take the upper of the two values of $\Phi(x)$ indicated.

3·075	0·9990	3·263	0·9994	3·731	0·99990	3·916	0·99995
3·105	0·9991	3·320	0·9995	3·759	0·99991	3·976	0·99996
3·138	0·9992	3·389	0·9996	3·791	0·99992	4·055	0·99997
3·174	0·9993	3·480	0·9997	3·826	0·99993	4·173	0·99998
3·215	0·9994	3·615	0·9998	3·867	0·99994	4·417	0·99999
			0·9999		0·99995		1·00000

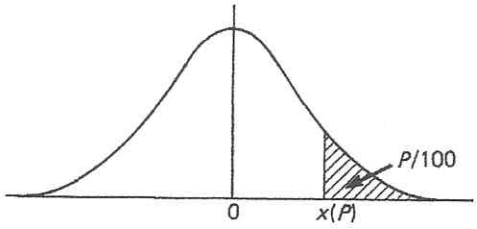
When $x > 3{\cdot}3$ the formula $1-\Phi(x) \doteqdot \frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}\left[1-\frac{1}{x^2}+\frac{3}{x^4}-\frac{15}{x^6}+\frac{105}{x^8}\right]$ is very accurate, with relative error less than $945/x^{10}$.

TABLE 5. PERCENTAGE POINTS OF THE NORMAL DISTRIBUTION

This table gives percentage points $x(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{\sqrt{2\pi}} \int_{x(P)}^{\infty} e^{-\frac{1}{2}t^2}\, dt.$$

If X is a variable, normally distributed with zero mean and unit variance, $P/100$ is the probability that $X \geqslant x(P)$. The lower P per cent points are given by symmetry as $-x(P)$, and the probability that $|X| \geqslant x(P)$ is $2P/100$.



P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$
50	0·0000	5·0	1·6449	3·0	1·8808	2·0	2·0537	1·0	2·3263	0·10	3·0902
45	0·1257	4·8	1·6646	2·9	1·8957	1·9	2·0749	0·9	2·3656	0·09	3·1214
40	0·2533	4·6	1·6849	2·8	1·9110	1·8	2·0969	0·8	2·4089	0·08	3·1559
35	0·3853	4·4	1·7060	2·7	1·9268	1·7	2·1201	0·7	2·4573	0·07	3·1947
30	0·5244	4·2	1·7279	2·6	1·9431	1·6	2·1444	0·6	2·5121	0·06	3·2389
25	0·6745	4·0	1·7507	2·5	1·9600	1·5	2·1701	0·5	2·5758	0·05	3·2905
20	0·8416	3·8	1·7744	2·4	1·9774	1·4	2·1973	0·4	2·6521	0·01	3·7190
15	1·0364	3·6	1·7991	2·3	1·9954	1·3	2·2262	0·3	2·7478	0·005	3·8906
10	1·2816	3·4	1·8250	2·2	2·0141	1·2	2·2571	0·2	2·8782	0·001	4·2649
5	1·6449	3·2	1·8522	2·1	2·0335	1·1	2·2904	0·1	3·0902	0·0005	4·4172

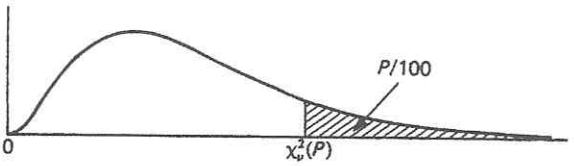
TABLE 8. PERCENTAGE POINTS OF THE χ^2-DISTRIBUTION

This table gives percentage points $\chi^2_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{2^{\nu/2}\,\Gamma\left(\frac{\nu}{2}\right)} \int_{\chi^2_\nu(P)}^{\infty} x^{\frac{1}{2}\nu-1}\, e^{-\frac{1}{2}x}\, dx.$$

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geqslant \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu-1}$ and unit variance.



(The above shape applies for $\nu \geqslant 3$ only. When $\nu < 3$ the mode is at the origin.)

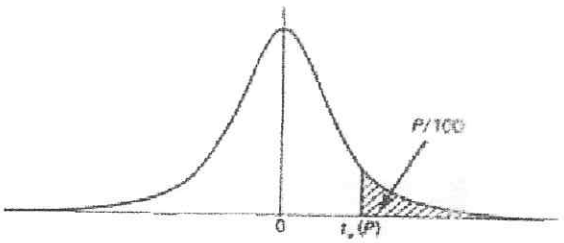
P	50	40	30	20	10	5	2·5	1	0·5	0·1	0·05
$\nu = 1$	0·4549	0·7083	1·074	1·642	2·706	3·841	5·024	6·635	7·879	10·83	12·12
2	1·386	1·833	2·408	3·219	4·605	5·991	7·378	9·210	10·60	13·82	15·20
3	2·366	2·946	3·665	4·642	6·251	7·815	9·348	11·34	12·84	16·27	17·73
4	3·357	4·045	4·878	5·989	7·779	9·488	11·14	13·28	14·86	18·47	20·00
5	4·351	5·132	6·064	7·289	9·236	11·07	12·83	15·09	16·75	20·52	22·11
6	5·348	6·211	7·231	8·558	10·64	12·59	14·45	16·81	18·55	22·46	24·10
7	6·346	7·283	8·383	9·803	12·02	14·07	16·01	18·48	20·28	24·32	26·02
8	7·344	8·351	9·524	11·03	13·36	15·51	17·53	20·09	21·95	26·12	27·87
9	8·343	9·414	10·66	12·24	14·68	16·92	19·02	21·67	23·59	27·88	29·67
10	9·342	10·47	11·78	13·44	15·99	18·31	20·48	23·21	25·19	29·59	31·42
11	10·34	11·53	12·90	14·63	17·28	19·68	21·92	24·72	26·76	31·26	33·14
12	11·34	12·58	14·01	15·81	18·55	21·03	23·34	26·22	28·30	32·91	34·82
13	12·34	13·64	15·12	16·98	19·81	22·36	24·74	27·69	29·82	34·53	36·48
14	13·34	14·69	16·22	18·15	21·06	23·68	26·12	29·14	31·32	36·12	38·11
15	14·34	15·73	17·32	19·31	22·31	25·00	27·49	30·58	32·80	37·70	39·72
16	15·34	16·78	18·42	20·47	23·54	26·30	28·85	32·00	34·27	39·25	41·31
17	16·34	17·82	19·51	21·61	24·77	27·59	30·19	33·41	35·72	40·79	42·88
18	17·34	18·87	20·60	22·76	25·99	28·87	31·53	34·81	37·16	42·31	44·43
19	18·34	19·91	21·69	23·90	27·20	30·14	32·85	36·19	38·58	43·82	45·97
20	19·34	20·95	22·77	25·04	28·41	31·41	34·17	37·57	40·00	45·31	47·50
21	20·34	21·99	23·86	26·17	29·62	32·67	35·48	38·93	41·40	46·80	49·01
22	21·34	23·03	24·94	27·30	30·81	33·92	36·78	40·29	42·80	48·27	50·51
23	22·34	24·07	26·02	28·43	32·01	35·17	38·08	41·64	44·18	49·73	52·00
24	23·34	25·11	27·10	29·55	33·20	36·42	39·36	42·98	45·56	51·18	53·48
25	24·34	26·14	28·17	30·68	34·38	37·65	40·65	44·31	46·93	52·62	54·95
26	25·34	27·18	29·25	31·79	35·56	38·89	41·92	45·64	48·29	54·05	56·41
27	26·34	28·21	30·32	32·91	36·74	40·11	43·19	46·96	49·64	55·48	57·86
28	27·34	29·25	31·39	34·03	37·92	41·34	44·46	48·28	50·99	56·89	59·30
29	28·34	30·28	32·46	35·14	39·09	42·56	45·72	49·59	52·34	58·30	60·73
30	29·34	31·32	33·53	36·25	40·26	43·77	46·98	50·89	53·67	59·70	62·16
32	31·34	33·38	35·66	38·47	42·58	46·19	49·48	53·49	56·33	62·49	65·00
34	33·34	35·44	37·80	40·68	44·90	48·60	51·97	56·06	58·96	65·25	67·80
36	35·34	37·50	39·92	42·88	47·21	51·00	54·44	58·62	61·58	67·99	70·59
38	37·34	39·56	42·05	45·08	49·51	53·38	56·90	61·16	64·18	70·70	73·35
40	39·34	41·62	44·16	47·27	51·81	55·76	59·34	63·69	66·77	73·40	76·09
50	49·33	51·89	54·72	58·16	63·17	67·50	71·42	76·15	79·49	86·66	89·56
60	59·33	62·13	65·23	68·97	74·40	79·08	83·30	88·38	91·95	99·61	102·7
70	69·33	72·36	75·69	79·71	85·53	90·53	95·02	100·4	104·2	112·3	115·6
80	79·33	82·57	86·12	90·41	96·58	101·9	106·6	112·3	116·3	124·8	128·3
90	89·33	92·76	96·52	101·1	107·6	113·1	118·1	124·1	128·3	137·2	140·8
100	99·33	102·9	106·9	111·7	118·5	124·3	129·6	135·8	140·2	149·4	153·2

TABLE 10. PERCENTAGE POINTS OF THE t-DISTRIBUTION

This table gives percentage points $t_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{\sqrt{\nu\pi}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu)} \int_{t_\nu(P)}^{\infty} \frac{dt}{(1+t^2/\nu)^{\frac{1}{2}(\nu+1)}}.$$

Let X_1 and X_2 be independent random variables having a normal distribution with zero mean and unit variance and a χ^2-distribution with ν degrees of freedom respectively; then $t = X_1/\sqrt{X_2/\nu}$ has Student's t-distribution with ν degrees of freedom, and the probability that $t \geqslant t_\nu(P)$ is $P/100$. The lower percentage points are given by symmetry as $-t_\nu(P)$, and the probability that $|t| \geqslant t_\nu(P)$ is $2P/100$.



The limiting distribution of t as ν tends to infinity is the normal distribution with zero mean and unit variance. When ν is large interpolation in ν should be harmonic.

P	40	30	25	20	15	10	5	2·5	1	0·5	0·1	0·05
$\nu = 1$	0·3249	0·7265	1·0000	1·3764	1·963	3·078	6·314	12·71	31·82	63·66	318·3	636·6
2	0·2887	0·6172	0·8165	1·0607	1·386	1·886	2·920	4·303	6·965	9·925	22·33	31·60
3	0·2767	0·5844	0·7649	0·9785	1·250	1·638	2·353	3·182	4·541	5·841	10·21	12·92
4	0·2707	0·5686	0·7407	0·9410	1·190	1·533	2·132	2·776	3·747	4·604	7·173	8·610
5	0·2672	0·5594	0·7267	0·9195	1·156	1·476	2·015	2·571	3·365	4·032	5·893	6·869
6	0·2648	0·5534	0·7176	0·9057	1·134	1·440	1·943	2·447	3·143	3·707	5·208	5·959
7	0·2632	0·5491	0·7111	0·8960	1·119	1·415	1·895	2·365	2·998	3·499	4·785	5·408
8	0·2619	0·5459	0·7064	0·8889	1·108	1·397	1·860	2·306	2·896	3·355	4·501	5·041
9	0·2610	0·5435	0·7027	0·8834	1·100	1·383	1·833	2·262	2·821	3·250	4·297	4·781
10	0·2602	0·5415	0·6998	0·8791	1·093	1·372	1·812	2·228	2·764	3·169	4·144	4·587
11	0·2596	0·5399	0·6974	0·8755	1·088	1·363	1·796	2·201	2·718	3·106	4·025	4·437
12	0·2590	0·5386	0·6955	0·8726	1·083	1·356	1·782	2·179	2·681	3·055	3·930	4·318
13	0·2586	0·5375	0·6938	0·8702	1·079	1·350	1·771	2·160	2·650	3·012	3·852	4·221
14	0·2582	0·5366	0·6924	0·8681	1·076	1·345	1·761	2·145	2·624	2·977	3·787	4·140
15	0·2579	0·5357	0·6912	0·8662	1·074	1·341	1·753	2·131	2·602	2·947	3·733	4·073
16	0·2576	0·5350	0·6901	0·8647	1·071	1·337	1·746	2·120	2·583	2·921	3·686	4·015
17	0·2573	0·5344	0·6892	0·8633	1·069	1·333	1·740	2·110	2·567	2·898	3·646	3·965
18	0·2571	0·5338	0·6884	0·8620	1·067	1·330	1·734	2·101	2·552	2·878	3·610	3·922
19	0·2569	0·5333	0·6876	0·8610	1·066	1·328	1·729	2·093	2·539	2·861	3·579	3·883
20	0·2567	0·5329	0·6870	0·8600	1·064	1·325	1·725	2·086	2·528	2·845	3·552	3·850
21	0·2566	0·5325	0·6864	0·8591	1·063	1·323	1·721	2·080	2·518	2·831	3·527	3·819
22	0·2564	0·5321	0·6858	0·8583	1·061	1·321	1·717	2·074	2·508	2·819	3·505	3·792
23	0·2563	0·5317	0·6853	0·8575	1·060	1·319	1·714	2·069	2·500	2·807	3·485	3·768
24	0·2562	0·5314	0·6848	0·8569	1·059	1·318	1·711	2·064	2·492	2·797	3·467	3·745
25	0·2561	0·5312	0·6844	0·8562	1·058	1·316	1·708	2·060	2·485	2·787	3·450	3·725
26	0·2560	0·5309	0·6840	0·8557	1·058	1·315	1·706	2·056	2·479	2·779	3·435	3·707
27	0·2559	0·5306	0·6837	0·8551	1·057	1·314	1·703	2·052	2·473	2·771	3·421	3·690
28	0·2558	0·5304	0·6834	0·8546	1·056	1·313	1·701	2·048	2·467	2·763	3·408	3·674
29	0·2557	0·5302	0·6830	0·8542	1·055	1·311	1·699	2·045	2·462	2·756	3·396	3·659
30	0·2556	0·5300	0·6828	0·8538	1·055	1·310	1·697	2·042	2·457	2·750	3·385	3·646
32	0·2555	0·5297	0·6822	0·8530	1·054	1·309	1·694	2·037	2·449	2·738	3·365	3·622
34	0·2553	0·5294	0·6818	0·8523	1·052	1·307	1·691	2·032	2·441	2·728	3·348	3·601
36	0·2552	0·5291	0·6814	0·8517	1·052	1·306	1·688	2·028	2·434	2·719	3·333	3·582
38	0·2551	0·5288	0·6810	0·8512	1·051	1·304	1·686	2·024	2·429	2·712	3·319	3·566
40	0·2550	0·5286	0·6807	0·8507	1·050	1·303	1·684	2·021	2·423	2·704	3·307	3·551
50	0·2547	0·5278	0·6794	0·8489	1·047	1·299	1·676	2·009	2·403	2·678	3·261	3·496
60	0·2545	0·5272	0·6786	0·8477	1·045	1·296	1·671	2·000	2·390	2·660	3·232	3·460
120	0·2539	0·5258	0·6765	0·8446	1·041	1·289	1·658	1·980	2·358	2·617	3·160	3·373
∞	0·2533	0·5244	0·6745	0·8416	1·036	1·282	1·645	1·960	2·326	2·576	3·090	3·291

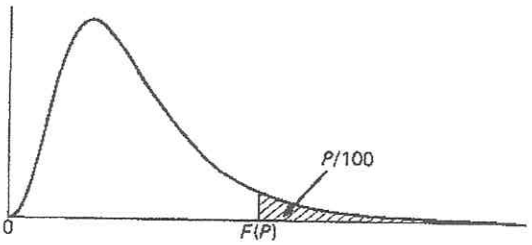
TABLE 12(*a*). 10 PER CENT POINTS OF THE *F*-DISTRIBUTION

The function tabulated is $F(P) = F(P|\nu_1, \nu_2)$ defined by the equation

$$\frac{P}{100} = \frac{\Gamma(\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2)}{\Gamma(\frac{1}{2}\nu_1)\,\Gamma(\frac{1}{2}\nu_2)}\,\nu_1^{\frac{1}{2}\nu_1}\,\nu_2^{\frac{1}{2}\nu_2}\int_{F(P)}^{\infty}\frac{F^{\frac{1}{2}\nu_1 - 1}}{(\nu_2 + \nu_1 F)^{\frac{1}{2}(\nu_1+\nu_2)}}\,dF,$$

for $P = 10$, 5, 2·5, 1, 0·5 and 0·1. The lower percentage points, that is the values $F'(P) = F'(P|\nu_1, \nu_2)$ such that the probability that $F \leqslant F'(P)$ is equal to $P/100$, may be found by the formula

$$F'(P|\nu_1, \nu_2) = 1/F(P|\nu_2, \nu_1).$$

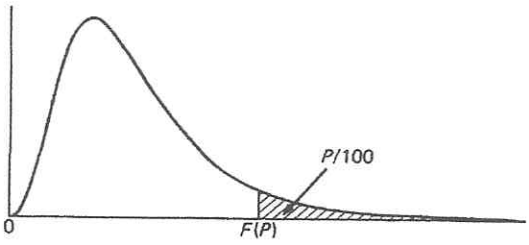


(This shape applies only when $\nu_1 \geqslant 3$. When $\nu_1 < 3$ the mode is at the origin.)

$\nu_1 =$	1	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 =$ 1	39·86	49·50	53·59	55·83	57·24	58·20	58·91	59·44	60·19	60·71	62·00	63·33
2	8·526	9·000	9·162	9·243	9·293	9·326	9·349	9·367	9·392	9·408	9·450	9·491
3	5·538	5·462	5·391	5·343	5·309	5·285	5·266	5·252	5·230	5·216	5·176	5·134
4	4·545	4·325	4·191	4·107	4·051	4·010	3·979	3·955	3·920	3·896	3·831	3·761
5	4·060	3·780	3·619	3·520	3·453	3·405	3·368	3·339	3·297	3·268	3·191	3·105
6	3·776	3·463	3·289	3·181	3·108	3·055	3·014	2·983	2·937	2·905	2·818	2·722
7	3·589	3·257	3·074	2·961	2·883	2·827	2·785	2·752	2·703	2·668	2·575	2·471
8	3·458	3·113	2·924	2·806	2·726	2·668	2·624	2·589	2·538	2·502	2·404	2·293
9	3·360	3·006	2·813	2·693	2·611	2·551	2·505	2·469	2·416	2·379	2·277	2·159
10	3·285	2·924	2·728	2·605	2·522	2·461	2·414	2·377	2·323	2·284	2·178	2·055
11	3·225	2·860	2·660	2·536	2·451	2·389	2·342	2·304	2·248	2·209	2·100	1·972
12	3·177	2·807	2·606	2·480	2·394	2·331	2·283	2·245	2·188	2·147	2·036	1·904
13	3·136	2·763	2·560	2·434	2·347	2·283	2·234	2·195	2·138	2·097	1·983	1·846
14	3·102	2·726	2·522	2·395	2·307	2·243	2·193	2·154	2·095	2·054	1·938	1·797
15	3·073	2·695	2·490	2·361	2·273	2·208	2·158	2·119	2·059	2·017	1·899	1·755
16	3·048	2·668	2·462	2·333	2·244	2·178	2·128	2·088	2·028	1·985	1·866	1·718
17	3·026	2·645	2·437	2·308	2·218	2·152	2·102	2·061	2·001	1·958	1·836	1·686
18	3·007	2·624	2·416	2·286	2·196	2·130	2·079	2·038	1·977	1·933	1·810	1·657
19	2·990	2·606	2·397	2·266	2·176	2·109	2·058	2·017	1·956	1·912	1·787	1·631
20	2·975	2·589	2·380	2·249	2·158	2·091	2·040	1·999	1·937	1·892	1·767	1·607
21	2·961	2·575	2·365	2·233	2·142	2·075	2·023	1·982	1·920	1·875	1·748	1·586
22	2·949	2·561	2·351	2·219	2·128	2·060	2·008	1·967	1·904	1·859	1·731	1·567
23	2·937	2·549	2·339	2·207	2·115	2·047	1·995	1·953	1·890	1·845	1·716	1·549
24	2·927	2·538	2·327	2·195	2·103	2·035	1·983	1·941	1·877	1·832	1·702	1·533
25	2·918	2·528	2·317	2·184	2·092	2·024	1·971	1·929	1·866	1·820	1·689	1·518
26	2·909	2·519	2·307	2·174	2·082	2·014	1·961	1·919	1·855	1·809	1·677	1·504
27	2·901	2·511	2·299	2·165	2·073	2·005	1·952	1·909	1·845	1·799	1·666	1·491
28	2·894	2·503	2·291	2·157	2·064	1·996	1·943	1·900	1·836	1·790	1·656	1·478
29	2·887	2·495	2·283	2·149	2·057	1·988	1·935	1·892	1·827	1·781	1·647	1·467
30	2·881	2·489	2·276	2·142	2·049	1·980	1·927	1·884	1·819	1·773	1·638	1·456
32	2·869	2·477	2·263	2·129	2·036	1·967	1·913	1·870	1·805	1·758	1·622	1·437
34	2·859	2·466	2·252	2·118	2·024	1·955	1·901	1·858	1·793	1·745	1·608	1·419
36	2·850	2·456	2·243	2·108	2·014	1·945	1·891	1·847	1·781	1·734	1·595	1·404
38	2·842	2·448	2·234	2·099	2·005	1·935	1·881	1·838	1·772	1·724	1·584	1·390
40	2·835	2·440	2·226	2·091	1·997	1·927	1·873	1·829	1·763	1·715	1·574	1·377
60	2·791	2·393	2·177	2·041	1·946	1·875	1·819	1·775	1·707	1·657	1·511	1·291
120	2·748	2·347	2·130	1·992	1·896	1·824	1·767	1·722	1·652	1·601	1·447	1·193
∞	2·706	2·303	2·084	1·945	1·847	1·774	1·717	1·670	1·599	1·546	1·383	1·000

TABLE 12(*b*). 5 PER CENT POINTS OF THE *F*-DISTRIBUTION

If $F = \frac{X_1}{\nu_1} \Big/ \frac{X_2}{\nu_2}$, where X_1 and X_2 are independent random variables distributed as χ^2 with ν_1 and ν_2 degrees of freedom respectively, then the probabilities that $F \geqslant F(P)$ and that $F \leqslant F'(P)$ are both equal to $P/100$. Linear interpolation in ν_1 and ν_2 will generally be sufficiently accurate except when either $\nu_1 > 12$ or $\nu_2 > 40$, when harmonic interpolation should be used.



(This shape applies only when $\nu_1 \geqslant 3$. When $\nu_1 < 3$ the mode is at the origin.)

$\nu_1 =$	1	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 = 1$	161·4	199·5	215·7	224·6	230·2	234·0	236·8	238·9	241·9	243·9	249·1	254·3
2	18·51	19·00	19·16	19·25	19·30	19·33	19·35	19·37	19·40	19·41	19·45	19·50
3	10·13	9·552	9·277	9·117	9·013	8·941	8·887	8·845	8·786	8·745	8·639	8·526
4	7·709	6·944	6·591	6·388	6·256	6·163	6·094	6·041	5·964	5·912	5·774	5·628
5	6·608	5·786	5·409	5·192	5·050	4·950	4·876	4·818	4·735	4·678	4·527	4·365
6	5·987	5·143	4·757	4·534	4·387	4·284	4·207	4·147	4·060	4·000	3·841	3·669
7	5·591	4·737	4·347	4·120	3·972	3·866	3·787	3·726	3·637	3·575	3·410	3·230
8	5·318	4·459	4·066	3·838	3·687	3·581	3·500	3·438	3·347	3·284	3·115	2·928
9	5·117	4·256	3·863	3·633	3·482	3·374	3·293	3·230	3·137	3·073	2·900	2·707
10	4·965	4·103	3·708	3·478	3·326	3·217	3·135	3·072	2·978	2·913	2·737	2·538
11	4·844	3·982	3·587	3·357	3·204	3·095	3·012	2·948	2·854	2·788	2·609	2·404
12	4·747	3·885	3·490	3·259	3·106	2·996	2·913	2·849	2·753	2·687	2·505	2·296
13	4·667	3·806	3·411	3·179	3·025	2·915	2·832	2·767	2·671	2·604	2·420	2·206
14	4·600	3·739	3·344	3·112	2·958	2·848	2·764	2·699	2·602	2·534	2·349	2·131
15	4·543	3·682	3·287	3·056	2·901	2·790	2·707	2·641	2·544	2·475	2·288	2·066
16	4·494	3·634	3·239	3·007	2·852	2·741	2·657	2·591	2·494	2·425	2·235	2·010
17	4·451	3·592	3·197	2·965	2·810	2·699	2·614	2·548	2·450	2·381	2·190	1·960
18	4·414	3·555	3·160	2·928	2·773	2·661	2·577	2·510	2·412	2·342	2·150	1·917
19	4·381	3·522	3·127	2·895	2·740	2·628	2·544	2·477	2·378	2·308	2·114	1·878
20	4·351	3·493	3·098	2·866	2·711	2·599	2·514	2·447	2·348	2·278	2·082	1·843
21	4·325	3·467	3·072	2·840	2·685	2·573	2·488	2·420	2·321	2·250	2·054	1·812
22	4·301	3·443	3·049	2·817	2·661	2·549	2·464	2·397	2·297	2·226	2·028	1·783
23	4·279	3·422	3·028	2·796	2·640	2·528	2·442	2·375	2·275	2·204	2·005	1·757
24	4·260	3·403	3·009	2·776	2·621	2·508	2·423	2·355	2·255	2·183	1·984	1·733
25	4·242	3·385	2·991	2·759	2·603	2·490	2·405	2·337	2·236	2·165	1·964	1·711
26	4·225	3·369	2·975	2·743	2·587	2·474	2·388	2·321	2·220	2·148	1·946	1·691
27	4·210	3·354	2·960	2·728	2·572	2·459	2·373	2·305	2·204	2·132	1·930	1·672
28	4·196	3·340	2·947	2·714	2·558	2·445	2·359	2·291	2·190	2·118	1·915	1·654
29	4·183	3·328	2·934	2·701	2·545	2·432	2·346	2·278	2·177	2·104	1·901	1·638
30	4·171	3·316	2·922	2·690	2·534	2·421	2·334	2·266	2·165	2·092	1·887	1·622
32	4·149	3·295	2·901	2·668	2·512	2·399	2·313	2·244	2·142	2·070	1·864	1·594
34	4·130	3·276	2·883	2·650	2·494	2·380	2·294	2·225	2·123	2·050	1·843	1·569
36	4·113	3·259	2·866	2·634	2·477	2·364	2·277	2·209	2·106	2·033	1·824	1·547
38	4·098	3·245	2·852	2·619	2·463	2·349	2·262	2·194	2·091	2·017	1·808	1·527
40	4·085	3·232	2·839	2·606	2·449	2·336	2·249	2·180	2·077	2·003	1·793	1·509
60	4·001	3·150	2·758	2·525	2·368	2·254	2·167	2·097	1·993	1·917	1·700	1·389
120	3·920	3·072	2·680	2·447	2·290	2·175	2·087	2·016	1·910	1·834	1·608	1·254
∞	3·841	2·996	2·605	2·372	2·214	2·099	2·010	1·938	1·831	1·752	1·517	1·000